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Homotopy perturbation method for a conservative $x^{1/3}$ force nonlinear oscillator

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A B S T R A C T

The homotopy perturbation method is used to obtain the periodic solutions of a conservative nonlinear oscillator for which the elastic force term is proportional to $x^{1/3}$. We find this method works very well for the whole range of initial amplitudes. Excellent agreement of the approximate frequencies with the exact ones has been demonstrated and discussed. Only one iteration leads to high accuracy of the solutions with a maximal relative error for the approximate frequency of less than 0.60% for small and large values of oscillation amplitude, while this relative error is as low as 0.050% for the second iteration. Comparison of the results obtained using this method with those obtained by different harmonic balance methods reveals that the former is more effective and convenient for these types of nonlinear oscillators.

1. Introduction

In recent years, considerable attention has been directed towards the study of strongly nonlinear oscillators, and several methods such as perturbation techniques [1–15] or harmonic balance based methods [15–23] have been used to find approximate solutions to nonlinear problems [1–4]. An excellent review of some asymptotic methods for strongly nonlinear equations can be found in detail in Refs. [1,24]. In general, given the nature of nonlinear phenomena, the approximate methods can only be applied within certain ranges of the physical parameters and to certain classes of problems [11].

The purpose of this paper is to calculate higher-order analytical approximations to the periodic solutions of a conservative nonlinear oscillator for which the elastic force term is proportional to $x^{1/3}$. To do this, we apply the homotopy perturbation method [24–41] derived by Ji-Huan He [24]. The work presented here extends previous results given in Beléndez et al. [41, 42] which relied primarily on a modified homotopy perturbation method as the tool for determining the oscillatory solutions for oscillatory systems with non-polynomial restoring forces. Excellent agreement of the approximate frequencies with the exact ones has been demonstrated and discussed, and we show that only two iterations lead to high accuracy of the solutions with a maximal relative error for the approximate frequency as low as 0.050%. As can be seen, the results presented in this paper reveal that the method is very effective and convenient for conservative nonlinear oscillators with non-polynomial elastic terms.

2. Solution procedure

We consider the following nonlinear oscillator

$$\frac{d^2x}{dt^2} + x^{1/3} = 0 \tag{1}$$

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with initial conditions

$$x(0) = A \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}t}(0) = 0 \tag{2}$$

which was introduced as a model "truly nonlinear oscillator" by Mickens [43–45] and has been studied by many other investigators [41,46–48]. Eq. (1) is a conservative nonlinear oscillator with a fractional power restoring force. We denote the angular frequency of these oscillations by ω and note that one of our major tasks is to determine ω (A), i.e., the functional behaviour of ω as a function of the initial amplitude.

Eq. (1) is not amenable to exact treatment and, therefore, approximate techniques must be resorted to. There is no small parameter in Eq. (1), so the standard perturbation methods cannot be applied directly. Since the homotopy perturbation method requires neither a small parameter nor a linear term in a differential equation, we can approximately solve Eq. (1) using the homotopy perturbation method. To do this, Eq. (1) is re-written in the form

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x = x - x^{1/3}. \tag{3}$$

For Eq. (3) we can establish the following homotopy

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} + x = p(x - x^{1/3}) \tag{4}$$

where p is the homotopy parameter. When $p = 0$, Eq. (4) becomes a linear differential equation for which an exact solution can be calculated; when $p = 1$, Eq. (4) then becomes the original problem. Now the homotopy parameter p is used to expand the solution $x(t)$ and the square of the unknown angular frequency ω as follows

$$x(t) = x_0(t) + px_1(t) + p^2x_2(t) + p^3x_3(t) + \cdots = \sum_{n=0}^{\infty} p^n x_n(t) \tag{5}$$

$$1 = \omega^2 - p\alpha_1 - p^2\alpha_2 - p^3\alpha_3 - \cdots = \omega^2 - \sum_{n=1}^{\infty} p^n \alpha_n \tag{6}$$

where α_i $(i = 1, 2, \ldots)$ are to be determined.

Substituting Eqs. (5) and (6) into Eq. (4) gives

$$\begin{aligned}&(x_0'' + px_1'' + p^2x_2'' + p^3x_3'' + \cdots) + (\omega^2 - p\alpha_1 - p^2\alpha_2 - p^3\alpha_3 - \cdots)(x_0 + px_1 + p^2x_2 + \cdots)\\ &\quad = p\left[(x_0 + px_1 + p^2x_2 + \cdots) - (x_0 + px_1 + p^2x_2 + p^3x_3 + \cdots)^{1/3}\right]\end{aligned} \tag{7}$$

and equating the terms with identical powers of p, we can obtain a series of linear equations, of which we write only the first three

$$x_0'' + \omega^2 x_0 = 0, \qquad x_0(0) = A, \qquad x_0'(0) = 0 \tag{8}$$

$$x_1'' + \omega^2 x_1 = (1 + \alpha_1)x_0 - x_0^{1/3}, \qquad x_1(0) = x_1'(0) = 0 \tag{9}$$

$$x_2'' + \omega^2 x_2 = \alpha_2 x_0 + (1 + \alpha_1)x_1 - \frac{1}{3}x_1 x_0^{-2/3}, \qquad x_2(0) = x_2'(0) = 0. \tag{10}$$

In Eqs. (8)–(10) we have taken into account the following expression [10]

$$\begin{aligned}f(x) &= f(x_0 + px_1 + p^2x_2 + \cdots)\\ &= f(x_0) + px_1 f'(x_0) + p^2\left[x_2 f'(x_0) + \frac{1}{2}x_1^2 f''(x_0)\right] + O(p^3).\end{aligned} \tag{11}$$

The solution of Eq. (8) is

$$x_0(t) = A\cos\omega t. \tag{12}$$

Substitution of this result into the right-hand side of Eq. (9) gives

$$x_1'' + \omega^2 x_1 = (1 + \alpha_1)A\cos\omega t - A^{1/3}\cos^{1/3}\omega t. \tag{13}$$

It is possible to do the following Fourier series expansion

$$A^{1/3}\cos^{1/3}\omega t = \sum_{n=0}^{\infty} a_{2n+1}\cos[(2n+1)\omega t] = a_1\cos\omega t + a_3\cos 3\omega t + \cdots \tag{14}$$

where

$$a_{2n+1} = \frac{4}{\pi}\int_0^{\pi/2} A^{1/3}\cos^{1/3}\theta\cos[(2n+1)\theta]\mathrm{d}\theta = \frac{2^{2/3}A^{1/3}\Gamma(4/3)}{\Gamma\left(\frac{2}{3}-n\right)\Gamma\left(\frac{5}{3}+n\right)}. \tag{15}$$

The first term of the expansion in Eq. (15) is given by

$$a_1 = \frac{4}{\pi}\int_0^{\pi/2} A^{1/3}\cos^{1/3}\theta\cos\theta\mathrm{d}\theta = \frac{3A^{1/3}\Gamma(7/6)}{\sqrt{\pi}\,\Gamma(2/3)} \tag{16}$$

where $\theta = \omega t$ and and $\Gamma(z)$ is the Euler gamma function [49]. This integral and another that will appear later were solved using symbolic software such as MATHEMATICA. Substituting Eq. (16) into Eq. (13), we have

$$x_1'' + \omega^2 x_1 = [(1+\alpha_1)A - a_1]\cos\omega t - \sum_{n=1}^{\infty} a_{2n+1}\cos[(2n+1)\omega t]. \tag{17}$$

No secular terms in $x_1(t)$ requires eliminating contributions proportional to $\cos\omega t$ on the right-hand side of Eq. (17)

$$(1+\alpha_1)A - a_1 = 0. \tag{18}$$

Substituting Eq. (16) into Eq. (18) and reordering, we can easily find that the solution for α_1 is

$$\alpha_1 = -1 + \frac{3\Gamma(7/6)}{A^{2/3}\sqrt{\pi}\,\Gamma(2/3)} = -1 + \frac{1.1596}{A^{2/3}}. \tag{19}$$

From Eqs. (6) and (19), writing $p = 1$, we can easily find that the first-order approximate frequency is

$$\omega_1(A) = \sqrt{\frac{3\Gamma(7/6)}{A^{2/3}\sqrt{\pi}\,\Gamma(2/3)}} = \frac{1.07685}{A^{1/3}}. \tag{20}$$

Now in order to obtain the correction term x_1 for the periodic solution x_0 we consider the following procedure. Taking into account Eqs. (17) and (18), we re-write Eq. (17) in the form

$$x_1'' + \omega^2 x_1 = -\sum_{n=1}^{\infty} a_{2n+1}\cos[(2n+1)\omega t] \tag{21}$$

with initial conditions $x_1(0) = 0$ and $x_1'(0) = 0$. The periodic solution to Eq. (21) can be written as follows

$$x_1(t) = \sum_{n=0}^{\infty} c_{2n+1}\cos[(2n+1)\omega t]. \tag{22}$$

Substituting Eq. (22) into Eq. (21) gives

$$-\omega^2\sum_{n=0}^{\infty} 4n(n+1)c_{2n+1}\cos[(2n+1)\omega t] = -\sum_{n=1}^{\infty} a_{2n+1}\cos[(2n+1)\omega t] \tag{23}$$

and then we can write the following expression for the coefficients c_{2n+1}

$$c_{2n+1} = \frac{a_{2n+1}}{4n(n+1)\omega^2} \tag{24}$$

for $n \geq 1$. Taking into account that $x_1(0) = 0$, Eq. (22) gives

$$c_1 = -\sum_{n=1}^{\infty} c_{2n+1} = \frac{11A^{1/3}\Gamma(4/3)\,{}_3F_2(\{1,1,4/3\},\{8/3,3\},-1)}{64\sqrt{3\pi}\,\Gamma(17/6)\omega^2} = \frac{0.025253A^{1/3}}{\omega^2}. \tag{25}$$

In this equation, ${}_pF_q(\{a_1, a_2, \dots, a_p\}, \{b_1, b_2, \dots, b_q\}, z)$ is the generalized hypergeometric function defined as follows [50]

$${}_pF_q(\{a_1, a_2, \dots, a_p\}, \{b_1, b_2, \dots, b_q\}, z) = \sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n}\frac{z^n}{n!} \tag{26}$$

where $(a)_n$ is the Pochhammer symbol defined as follows

$$(a)_n = a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}. \tag{27}$$

To determine the second-order approximate solution we substitute Eqs. (12) and (22) into Eq. (9) and we obtain

$$x_2'' + \omega^2 x_2 = \alpha_2 A \cos\omega t + (1+\alpha_1)\sum_{n=0}^{\infty} c_{2n+1}\cos[(2n+1)\omega t] - \frac{1}{3}A^{-2/3}\cos^{-2/3}\omega t \sum_{n=0}^{\infty} c_{2n+1}\cos[(2n+1)\omega t] \tag{28}$$

which can be written as follows

$$\begin{aligned} x_2'' + \omega^2 x_2 =\ & \alpha_2 A \cos\omega t + (1+\alpha_1)\sum_{n=0}^{\infty} c_{2n+1}\cos[(2n+1)\omega t] \\ & - \frac{1}{3}c_1 A^{-2/3}\cos^{1/3}\omega t - \frac{1}{3}A^{-2/3}\cos^{-2/3}\omega t \sum_{n=1}^{\infty} c_{2n+1}\cos[(2n+1)\omega t]. \end{aligned} \tag{29}$$

We consider the Fourier series expansions

$$\frac{1}{3}c_1 A^{-2/3}\cos^{1/3}\omega t = \sum_{m=0}^{\infty} d_{2m+1}\cos[(2m+1)\omega t] = d_1\cos\omega t + d_3\cos 3\omega t + \cdots \tag{30}$$

$$\frac{1}{3}A^{-2/3}\cos^{-2/3}\omega t \sum_{n=1}^{\infty} c_{2n+1}\cos[(2n+1)\omega t] = \sum_{m=0}^{\infty} e_{2m+1}\cos[(2m+1)\omega t] = e_1\cos\omega t + e_3\cos 3\omega t + \cdots. \tag{31}$$

The first term of the Fourier series expansion in Eq. (30) is given by

$$\begin{aligned} d_1 &= \frac{4}{\pi}\sum_{n=0}^{\infty}\int_0^{\pi/2}\frac{1}{3}c_{2n+1}A^{-2/3}\cos^{1/3}\theta\cos\theta\,\mathrm{d}\theta = \frac{2^{2/3}\Gamma(4/3)c_1}{3A^{2/3}\Gamma(2/3)\Gamma(5/3)} \\ &= \frac{11\,\Gamma^2(4/3)\,{}_3F_2(\{1,1,4/3\},\{8/3,3\},-1)}{96\sqrt{3\pi}\,A^{1/3}2^{1/3}\Gamma(2/3)\Gamma(5/3)\Gamma(17/6)\omega^2} = \frac{0.00976107}{A^{1/3}\omega^2} \end{aligned} \tag{32}$$

and the first term of the expansion in Eq. (31) is given by

$$\begin{aligned} e_1 &= \frac{4}{\pi}\sum_{n=1}^{\infty}\int_0^{\pi/2}\frac{1}{3}c_{2n+1}A^{-2/3}\cos^{-2/3}\theta\cos[(2n+1)]\cos\theta\,\mathrm{d}\theta \\ &= \sum_{n=1}^{\infty}\frac{2^{2/3}\Gamma(4/3)c_{2n+1}}{3A^{2/3}\Gamma\left(\frac{2}{3}-n\right)\Gamma\left(\frac{5}{3}+n\right)} = \sum_{n=1}^{\infty}\frac{\Gamma^2(4/3)}{3A^{1/3}2^{2/3}n(n+1)\Gamma^2\left(\frac{2}{3}-n\right)\Gamma^2\left(\frac{5}{3}+n\right)\omega^2} \\ &= \frac{\Gamma^2(4/3)\,{}_4F_3(\{1,1,4/3,4/3\},\{8/3,8/3,3\},1)}{128\pi\,A^{1/3}2^{2/3}\Gamma^2(11/6)\omega^2} = \frac{0.00248656}{A^{1/3}\omega^2}. \end{aligned} \tag{33}$$

Substituting Eqs. (30) and (31) into Eq. (29) and eliminating the secular terms we obtain

$$\alpha_2 A + (1+\alpha_1)c_1 - d_1 - e_1 = 0. \tag{34}$$

Substituting Eqs. (19), (25), (32) and (33) into Eq. (34), and reordering, Eq. (34) can be solved for α_2, that is

$$\alpha_2 = \frac{1}{A}[-(1+\alpha_1)c_1 + d_1 + e_1] = -\frac{0.0170356}{A^{4/3}\omega^2}. \tag{35}$$

From Eqs. (6), (19) and (35), and taking $p = 1$, one can easily obtain the following expression for the second-order approximate frequency

$$\omega_2(A) = \frac{1.06991}{A^{1/3}}. \tag{36}$$

3. Comparison with the exact and other approximate solutions

We illustrate the accuracy of the modified approach by comparing the approximate solutions previously obtained with the exact frequency ω_{ex} and other results in the literature. In particular, we will consider the solution of Eq. (1) using the harmonic balance method [43], and a modified homotopy perturbation method [41].

The exact frequency is given by the following expression [41]

$$\omega_{ex}(A) = \frac{2\pi\,\Gamma(5/4)}{\sqrt{6}\,\Gamma(3/4)\Gamma(1/2)A^{1/3}} = \frac{1.070451}{A^{1/3}}. \tag{37}$$

Table 1
Comparison of the exact and approximate frequencies obtained using different methods.

	HPM (this paper)	HBM [43]	HBM [46]	HBM [47]	MHPM [41]
$A^{1/3}\omega_1$ (% error)	1.07685 (0.60%)	1.04912 (2.0%)	1.07685 (0.60%)	1.07685 (0.60%)	1.07685 (0.60%)
$A^{1/3}\omega_2$ (% error)	1.06991 (0.050%)	1.06341 (0.66%)	1.06928 (0.11%)	1.06922 (0.12%)	1.06861 (0.17%)

Applying the first and second approximations based on the exact harmonic balance method to the equation

$$\left(\frac{\mathrm{d}^2x}{\mathrm{d}t^2}\right)^3 + x = 0. \tag{38}$$

Mickens obtained the following expressions for the frequency [43]

$$\omega_{M1}(A) = \frac{1.04912}{A^{1/3}} \quad \text{Relative error} = 2.0\% \tag{39}$$

$$\omega_{M2}(A) = \frac{1.06341}{A^{1/3}} \quad \text{Relative error} = 0.66\%. \tag{40}$$

Lim and Wu approximately solved Eq. (1) by using an improved harmonic balance method in which linearization is carried out prior to harmonic balancing [46]. They achieved the following results for the first and the second-order approximations

$$\omega_{LW1}(A) = \frac{1.07685}{A^{1/3}} \quad \text{Relative error} = 0.6\% \tag{41}$$

$$\omega_{LW2}(A) = \frac{1.06928}{A^{1/3}} \quad \text{Relative error} = 0.11\%. \tag{42}$$

Wu, Sun and Lim [47] also approximately solved Eq. (1) using another improved harmonic balance method that incorporates salient features of both Newton's method and the harmonic balance method. They achieved the following results for the first- and second-order approximations

$$\omega_{WSL1}(A) = \frac{1.07685}{A^{1/3}} \quad \text{Relative error} = 0.60\% \tag{43}$$

$$\omega_{WSL2}(A) = \frac{1.06922}{A^{1/3}} \quad \text{Relative error} = 0.12\%. \tag{44}$$

Beléndez et al. [41] also approximately solved Eq. (1) using a modified homotopy perturbation method. As we have seen in this paper, applying the standard homotopy perturbation method to this oscillator, an infinite series is obtained for the first analytical approximate solution and this series must be introduced in the linear differential equation to obtain the second-order approximate solution. However, since it is difficult to work with an infinite series, they truncated this series before solving the subsequent linear differential equation considering only two harmonics for the second-order approximation, three harmonics for the third-order approximation, and so on. In this sense, the truncated approximate solutions have the same form as those considered when harmonic balance methods are applied. They achieved the following results for the first and second approximation orders

$$\omega_{B1}(A) = \frac{1.07685}{A^{1/3}} \quad \text{Relative error} = 0.60\% \tag{45}$$

$$\omega_{B2}(A) = \frac{1.06861}{A^{1/3}} \quad \text{Relative error} = 0.17\%. \tag{46}$$

The approximate frequency values and their relative errors obtained in this paper applying the homotopy perturbation method are the following (Eqs. (20) and (36))

$$\omega_1(A) = \frac{1.07685}{A^{1/3}} \quad \text{Relative error} = 0.60\% \tag{47}$$

$$\omega_2(A) = \frac{1.06991}{A^{1/3}} \quad \text{Relative error} = 0.050\%. \tag{48}$$

In Table 1 we present the comparison between the approximate and exact frequencies for the second-order approximation using different methods. It is clear that for the second-order approximation, the results obtained in this paper are better than those obtained previously by other authors.

4. Conclusions

The homotopy perturbation method has been used to obtain two approximate frequencies for a conservative nonlinear oscillatory system for which the elastic force term is proportional to $x^{1/3}$. Excellent agreement between approximate frequencies and the exact one has been demonstrated and discussed, and the discrepancy between the second-order approximate frequency, $\omega_2(A)$, and the exact one is as low as 0.050%. This is the best second-order approximate frequency for this oscillatory system reported in the literature. I believe that the homotopy perturbation method has great potential and can be applied to other strongly nonlinear oscillators with non-polynomial terms.

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References

[1] J.H. He, Non-perturbative methods for strongly nonlinear problems, Dissertation, De-Verlag im Internet GmbH, Berlin, 2006.
[2] A.H. Nayfeh, Problems in Perturbations, Wiley, New York, 1985.
[3] J.H. He, A review on some new recently developed nonlinear analytical techniques, Int. J. Nonlinear Sci. Numer. Simul. 1 (2000) 51–70.
[4] R.E. Mickens, Oscillations in Planar Dynamics Systems, World Scientific, Singapore, 1996.
[5] J.H. He, A new perturbation technique which is also valid for large parameters, J. Sound Vibration 229 (2000) 1257–1263.
[6] J.H. He, Modified Lindstedt–Poincare methods for some non-linear oscillations. Part III: Double series expansion, Int. J. Nonlinear Sci. Numer. Simul. 2 (2001) 317–320.
[7] J.H. He, Modified Lindstedt–Poincare methods for some non-linear oscillations. Part I: Expansion of a constant, Int. J. Nonlinear Mech. 37 (2002) 309–314.
[8] J.H. He, Modified Lindstedt–Poincare methods for some non-linear oscillations. Part II: A new transformation, Int. J. Nonlinear Mech. 37 (2002) 315–320.
[9] A. Beléndez, T. Beléndez, A. Hernández, S. Gallego, M. Ortuño, C. Neipp, Comments on investigation of the properties of the period for the nonlinear oscillator $\ddot{x} + (1 + \dot{x}^2)x = 0$, J. Sound Vibration 303 (2007) 937–942.
[10] P. Amore, A. Aranda, Improved Lindstedt–Poincaré method for the solution of nonlinear problems, J. Sound Vibration 283 (2005) 1115–1136.
[11] P. Amore, F.M. Fernández, Exact and approximate expressions for the period of anharmonic oscillators, Eur. J. Phys. 26 (2005) 589–601.
[12] J.H. He, Homotopy perturbation method for bifurcation on nonlinear problems, Int. J. Nonlinear Sci. Numer. Simul. 6 (2005) 207–208.
[13] P. Amore, A. Raya, F.M. Fernández, Alternative perturbation approaches in classical mechanics, Eur. J. Phys. 26 (2005) 1057–1063.
[14] P. Amore, A. Raya, F.M. Fernández, Comparison of alternative improved perturbative methods for nonlinear oscillations, Phys. Lett. A 340 (2005) 201–208.
[15] D.H. Shou, J.H. He, Application of parameter-expanding method to strongly nonlinear oscillators, Int. J. Nonlinear Sci. Numer. Simul. 8 (1) (2007) 121–124.
[16] R.E. Mickens, Mathematical and numerical study of the Duffing-harmonic oscillator, J. Sound Vibration 244 (2001) 563–567.
[17] H.P.W. Gottlieb, Harmonic balance approach to limit cycles for nonlinear jerk equations, J. Sound Vibration 297 (2006) 243–250.
[18] C.W. Lim, B.S.Wu. Sun, Higher accuracy analytical approximations to the Duffing-harmonic oscillator, J. Sound Vibration 296 (2006) 1039–1045.
[19] A. Beléndez, A. Hernández, A. Márquez, T. Beléndez, C. Neipp, Analytical approximations for the period of a simple pendulum, Eur. J. Phys 27 (2006) 539–551.
[20] A. Beléndez, A. Hernández, T. Beléndez, M.L. Álvarez, S. Gallego, M. Ortuño, C. Neipp, Application of the harmonic balance method to a nonlinear oscillator typified by a mass attached to a stretched wire, J. Sound Vibration 302 (2007) 1018–1029.
[21] H. Hu, J.H. Tang, Solution of a Duffing-harmonic oscillator by the method of harmonic balance, J. Sound Vibration 294 (2006) 637–639.
[22] H. Hu, Solution of a quadratic nonlinear oscillator by the method of harmonic balance, J. Sound Vibration 293 (2006) 462–468.
[23] G.R. Itovich, J.L. Moiola, On period doubling bifurcations of cycles and the harmonic balance method, Chaos, Solitons Fractals 27 (2005) 647–665.
[24] J.H. He, Some asymptotic methods for strongly nonlinear equations, Internat. J. Modern Phys. B 20 (2006) 1141–1199.
[25] J.H. He, New interpretation of homotopy perturbation method, Internat. J. Modern Phys. B 20 (2006) 2561–2568.
[26] J.H. He, The homotopy perturbation method for nonlinear oscillators with discontinuities, Appl. Math. Comput. 151 (2004) 287–292.
[27] M.S.H. Chowdhury, I. Hashim, Application of homotopy-perturbation method to nonlinear population dynamics models, Phys. Lett. A 368 (2007) 251–258.
[28] F. Shakeri, M. Dehghan, Inverse problem of diffusion by He's homotopy perturbation method, Phys. Scr. 75 (2007) 551–556.
[29] A. Beléndez, C. Pascual, E. Fernández, C. Neipp, T. Beléndez, Higher-order approximate solutions to the relativistic and Duffing-harmonic oscillators by modified He's homotopy methods, Phys. Scr. 77 (2008) 025004.
[30] A. Beléndez, A. Hernández, T. Beléndez, A. Márquez, Application of the homotopy perturbation method to the nonlinear pendulum, Eur. J. Phys. 28 (2007) 93–104.
[31] A. Beléndez, A. Hernández, T. Beléndez, E. Fernández, M.L. Álvarez, C. Neipp, Application of He's homotopy perturbation method to the Duffing-harmonic oscillator, Int. J. Nonlinear Sci. Numer. Simul. 8 (1) (2007) 79–88.
[32] A. Beléndez, C. Pascual, A. Márquez, D.I. Méndez, Application of He's homotopy perturbation method to the relativistic (an)harmonic oscillator. I: Comparison between approximate and exact frequencies, Int. J. Nonlinear Sci. Numer. Simul. 8 (4) (2007) 483–491.
[33] A. Beléndez, C. Pascual, D.I. Méndez, M.L. Álvarez, C. Neipp, Application of He's homotopy perturbation method to the relativistic (an)harmonic oscillator. II: A more accurate approximate solution, Int. J. Nonlinear Sci. Numer. Simul. 8 (4) (2007) 493–504.
[34] L.-N. Zhang, J.H. He, Homotopy perturbation method for the solution of the electrostatic potential differential equation, Math. Prob. Engng. 2006 (2006) 1–6.
[35] D.D. Ganji, A. Sadighi, Application of He's homotopy-perturbation method to nonlinear coupled systems of reaction-diffusion equations, Int. J. Nonlinear Sci. Numer. Simul. 7 (4) (2006) 411–418.
[36] A. Siddiqui, R. Mahmood, Q. Ghori, Thin film flow of a third grade fluid on moving a belt by He's homotopy perturbation method, Int. J. Nonlinear Sci. Numer. Simul. 7 (1) (2006) 15–26.
[37] J.H. He, Homotopy perturbation method for solving boundary value problems, Phys. Lett. A 350 (2006) 87–88.
[38] M. Rafei, D.D. Ganji, Explicit solutions of Helmhotlz equation and fifth-order KdV equation using homotopy perturbation method, Int. J. Nonlinear Sci. Numer. Simul. 7 (3) (2006) 321–328.
[39] D.D. Ganji, The application of He's homotopy perturbation method to nonlinear equations arising in heat transfer, Phys. Lett. 355 (2006) 337–341.
[40] P.D. Ariel, T. Hayat, Homotopy perturbation method and axisymmetric flow over a stretching sheet, Int. J. Nonlinear Sci. Numer. Simul. 7 (4) (2006) 399–406.

[41] A. Beléndez, C. Pascual, S. Gallego, M. Ortuño, C. Neipp, Application of a modified He's homotopy perturbation method to obtain higher-order approximations of an $x^{1/3}$ force nonlinear oscillator, Phys. Lett. A 371 (2007) 421–426.
[42] A. Beléndez, M.L. Alvarez, D.I. Méndez, E. Fernández, M.S. Yebra, T. Beléndez, Approximate solutions for conservative nonlinear oscillators by He's homotopy method, Z. Nat. forsch. : Phys. Sci. 63a (2008) 529–537.
[43] R.E. Mickens, Analysis of non-linear oscillators having non-polynomial elastic terms, J. Sound Vibration 255 (2002) 789–792.
[44] R.E. Mickens, Oscillations in an $x^{4/3}$ potential, J. Sound Vibration 246 (2001) 375–378.
[45] K. Cooper, R.E. Mickens, Generalized harmonic balance/numerical method for determining analytical approximations to the periodic solutions of the $x^{4/3}$ potential, J. Sound Vibration 250 (2002) 951–954.
[46] C.W. Lim, B.S. Wu., Accurate higher-order approximations to frequencies of nonlinear oscillators with fractional powers, J. Sound Vibration 281 (2005) 1157–1162.
[47] B.S. Wu., W.P. Sun, C.W. Lim, An analytical approximate technique for a class of strongly non-linear oscillators, Int. J. Nonlinear Mech. 41 (2006) 766–774.
[48] T. Ozis, A. Yildirim, Determination of periodic solution for a $u^{1/3}$ force by He's modified Lindstedt–Poincaré method, J. Sound Vibration 301 (2007) 415–419.
[49] G. Arfken, The incomplete gamma function and related functions, in: Mathematical Methods in Physics, Academic Press, Orlando FL, 1985, pp. 565–572. (Section 10.5).
[50] F. Oberhettinger, Hypergeometric functions, in: M. Abramowitz, I.A. Stegun (Eds.), Handbook of Mathematical Functions, Dover Publications Inc, New York, 1972.